Analytic Solutions II

- Gray Atmosphere
- Radiative equilibrium
- Temperature structure ⇔ Limb darkening

Why Do Gray Atmosphere?

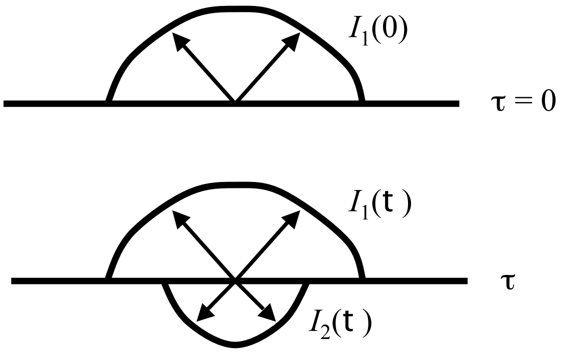
- Opacity independent of frequency
- Only true gray opacity is electron scattering
- Demonstrates radiative equilibrium
- Gives first guess at temperature structure
- Can relate to more general/realistic situations
- Can get exact solution: test approximate numerical techniques
- Can set exam questions…

Eddington Two Stream Approach

Crude representation of radiation field:

$$I(\tau,\mu) = I_1(\tau) \quad \mu > 0$$
$$\quad\quad = I_2(\tau) \quad \mu < 0$$

I_1 and I_2 are constant at any given τ
Boundary condition: $I_2(\tau = 0) = 0$ (no incident radiation)



Substitute I_1 and I_2 into integral expressions for J, H, K:

$$J(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\,\mathrm{d}\mu = \frac{1}{2}\int_{0}^{1} I_1(\tau)\,\mathrm{d}\mu + \frac{1}{2}\int_{-1}^{0} I_2(\tau)\,\mathrm{d}\mu = \frac{1}{2}[I_1(\tau) + I_2(\tau)]$$

$$H(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\mu\,\mathrm{d}\mu = \frac{1}{4}[I_1(\tau) - I_2(\tau)]$$

$$K(\tau) = \frac{1}{2}\int_{-1}^{1} I(\tau,\mu)\mu^2\,\mathrm{d}\mu = \frac{1}{6}[I_1(\tau) + I_2(\tau)]$$

Equations for J and K: first Eddington approximation:

$$J(\tau) = 3K(\tau)$$

Surface condition $I_2(\tau = 0) = 0$: second Eddington approx:

$$J(0) = I_1(0)/2,\ H(0) = I_1(0)/4 \Rightarrow J(0) = 2H(0)$$

Major assumption in gray atmosphere: opacity is independent of frequency: $d\kappa_\nu / d\nu = 0$. ERT simplifies considerably since we can integrate over frequency.

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I - S$$

where

$$I = \int_0^\infty I_\nu \, d\nu \quad ; \quad S = \int_0^\infty S_\nu \, d\nu$$

Now assume:

1. Opacity independent of frequency (see above)
2. Radiative equilibrium: Total radiative flux is constant throughout atmosphere: $dF / d\tau = 0$
3. Atmosphere is in LTE: $S_\nu(T) = B_\nu(T)$ and $S(T) = B(T)$
   $S(T)$ isotropic

ERT: Moment Equations

ERT complex => approximations => semi-analytic solutions
Assume S_ν is isotropic => form *moment equations*

First, perform angle averaging, apply: $\frac{1}{4\pi}\int d\Omega = \frac{1}{2}\int d\mu$

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

=>

$$\frac{1}{2}\int_{-1}^{1} \mu \frac{dI_\nu}{d\tau_\nu} d\mu = \frac{1}{2}\int_{-1}^{1} I_\nu d\mu - \frac{1}{2}\int_{-1}^{1} S_\nu d\mu$$

$$\frac{dH_\nu(\tau_\nu)}{d\tau_\nu} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

$$-\frac{dH_\nu(z)}{dz} = \kappa_\nu \rho J_\nu(z) - \kappa_\nu \rho S_\nu(z)$$

Multiply ERT by μ and then applying angle averaging:

$$\frac{1}{2}\int_{-1}^{1}\mu^2\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu}\mathrm{d}\mu = \frac{1}{2}\int_{-1}^{1}\mu\, I_\nu\,\mathrm{d}\mu - \frac{1}{2}\int_{-1}^{1}\mu\, S_\nu\,\mathrm{d}\mu$$

$$\frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = H_\nu(\tau_\nu)$$

= 0 for isotropic S_ν

Substitute into previous moment equation:

$$\frac{\mathrm{d}^2K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

Moment equations. Be able to derive these.

$$\frac{\mathrm{d}H_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

$$\frac{\mathrm{d}K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu} = H_\nu(\tau_\nu)$$

$$\frac{\mathrm{d}^2K_\nu(\tau_\nu)}{\mathrm{d}\tau_\nu^2} = J_\nu(\tau_\nu) - S_\nu(\tau_\nu)$$

Using frequency integrated terms, ERT moment equations become:

$$\frac{dH(\tau)}{d\tau} = J(\tau) - S(\tau)$$
$$\frac{dK(\tau)}{d\tau} = H(\tau)$$

Radiative equilibrium => flux is constant throughout atmosphere, so $dH / d\tau = 0$, since $H = F / 4$.

This gives $S(\tau) = J(\tau)$ and $K(\tau) = H(\tau + q)$, since $H(\tau) = H$ (constant) and q is a constant of integration.

Using the Eddington approximation, $J(\tau) = 3K(\tau)$, we get

$$J(\tau) = 3H(\tau + q)$$

Using the second Eddington approximation, $J(\tau = 0) = 2H$, we get $q = 2/3$, which gives the Milne-Eddington approximation for a gray atmosphere:

$$S(\tau) \approx \frac{3}{4}\left(\tau + \frac{2}{3}\right)F$$

since $F = 4H$ and $S = J$. Here $F = B = (\sigma/\pi)T_{eff}^4$

The exact solution is usually written:

$$S(\tau) = \frac{3}{4}[\tau + q(\tau)]F$$

where $q(\tau)$ is the *Hopf function* which varies slowly with τ

Gray Atmosphere Temperature Structure:

LTE gives $S(\tau) = B(\tau) = (\sigma/\pi)\ T(\tau)^4$, and using the Milne-Eddington approx above, we get

$$T(\tau) \approx \left(\frac{3}{4}\tau + \frac{1}{2}\right)^{1/4} T_{\mathrm{eff}}$$

This gives $T_{\mathrm{eff}} = T(\tau = 2/3)$ as it should

Gray Atmosphere Limb Darkening:

Using the Eddington-Barbier surface relation $I(0,\mu) = S(\tau = \mu)$ gives $I(0,\mu) = 3/4F(\mu + 2/3)$, so the centre-to-limb variation for a gray star is:

$$\frac{I(0,\mu)}{I(0,1)} = \frac{3}{5}(\mu + 2/3)$$

This gives $I(0,0) / I(0,1) = 0.4$, which is in excellent agreement with observed Solar limb darkening

Shows Sun in radiative (not convective) equilibrium (Schwarzschild 1906)